MATH 141: Some Practice Final Problems

Here are problems that cover the last two weeks of our class.

The final is cumulative; you should look at Practice Midterm 1+2 and Midterm 1+2 as well.

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1. Compute this integral:

$$\int_0^1 \left(\sum_{i=1}^{6} (i+1)x^i \right) dx$$

$$= \int_0^1 \left(2x^1 + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6\right) dx$$

$$= 2\frac{x^2}{2} + 3\frac{x^3}{3} + 4\frac{x^4}{4} + 5\frac{x^5}{5} + 6\frac{x^6}{6} + 7\frac{x^7}{7} \Bigg|_0^1$$

$$= x^2 + x^3 + x^4 + x^5 + x^6 + x^7 \Big|_0^1$$

$$= 1^2 + 1^3 + 1^4 + 1^5 + 1^6 + 1^7$$
$$- \left(0^2 + 0^3 + 0^4 + 0^5 + 0^6 + 0^7\right)$$

$$= \boxed{6}$$

2. Evaluate the following expressions. If applicable, you are allowed to use Fundamental Theorem of Calculus.

(a) $\sum_{i=1}^{5} \frac{f(i)}{i}$ given that $f(x) = x^2$

$$= \frac{f(1)}{1} + \frac{f(2)}{2} + \frac{f(3)}{3} + \frac{f(4)}{4} + \frac{f(5)}{5}$$

$$= \frac{1^2}{1} + \frac{2^2}{2} + \frac{3^2}{3} + \frac{4^2}{4} + \frac{5^2}{5}$$

$$= 1 + 2 + 3 + 4 + 5 = \boxed{15}$$

(b) $\frac{d}{d\theta} \int_{-\pi}^{\theta} \sin x \, dx$

$= \boxed{\sin\theta}$ by FTC Part 1, $\sin\theta$ is continuous on $\mathbb{R}$

(c) $\int_1^5 \frac{x^2 - 1}{x - 1} dx = \int_1^5 \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} dx$

$$= \int_1^5 (x+1) \, dx$$

$$= \frac{x^2}{2} + x \Big|_1^5$$

$$= \frac{5^2}{2} + 5 - \left(\frac{1^2}{2} + 1\right) = \boxed{16}$$

largest power in denom is $\sqrt{x^4} = x^2$

(d) $\lim_{x\to\infty} \frac{x+1}{\sqrt{x^4-2}}$

$$= \lim_{x\to\infty} \frac{\frac{x+1}{x^2}}{\frac{\sqrt{x^4-2}}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{\sqrt{\frac{x^4-2}{x^4}}}$$

$$= \frac{\lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{1}{x^2}}{\sqrt{\lim_{x\to\infty} 1 - \lim_{x\to\infty}\frac{2}{x^4}}} = \frac{0+0}{\sqrt{1+0}} = \boxed{0}$$

(e) $\int (\sec(x)\tan(x) - \csc(x)\cot(x))\, dx$

$$= \int \sec(x)\tan(x)\,dx - \int \csc(x)\cot(x)\,dx$$

$$= \sec(x) - (-\csc(x)) + C$$

$$= \boxed{\sec(x) + \csc(x) + C}$$

(f) $\lim_{x\to\infty}(x^5 - x^3) = \infty - \infty$ indeterminate form

$$\lim_{x\to\infty} x^5 - x^3 = \lim_{x\to\infty} x^3(x^2-1)$$

$$= \infty \cdot \infty$$

$$= \boxed{\infty}$$

(g) $\displaystyle\int_{-1}^{1} \frac{3x^2+4x+4}{x}\,dx$

Domain is $(-\infty, 0) \cup (0, \infty)$, hence discontinuous at 0.

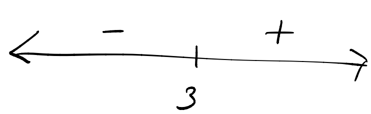
We are trying to integrate on $[-1, 1]$.

FTC requires continuity on $[-1, 1]$ to evaluate a definite integral.

$\boxed{\text{Answer: we are unable to solve with Calculus I techniques.}}$

(h) $\displaystyle\int_{0}^{6} |x-3|\,dx$

Sign diagram for $x-3$



So $\displaystyle\int_0^6 |x-3|\,dx = \int_0^3 |x-3|\,dx + \int_3^6 |x-3|\,dx$

$$= \int_0^3 -(x-3)\,dx + \int_3^6 (x-3)\,dx$$

$$= -\frac{x^2}{2} + 3x \Big|_0^3 + \left[\frac{x^2}{2} - 3x\right]_3^6$$

$$= -\frac{3^2}{2} + 3\cdot 3 - \left(-\frac{0^2}{2} + 3\cdot 0\right) + \frac{6^2}{2} - 3\cdot 6 - \left(\frac{3^2}{2} - 3\cdot 3\right)$$

$$= -\frac{9}{2} + 9 + 18 - 18 + \frac{9}{2}$$

$$= \boxed{9}$$

3. Suppose $f(x) = x^2$. Approximate the area underneath the curve on the interval $[1, 2]$ using four rectangles and right endpoints.

   Only set up the sum; do not compute it.

$n = 4$, $a = 1$, $b = 2$

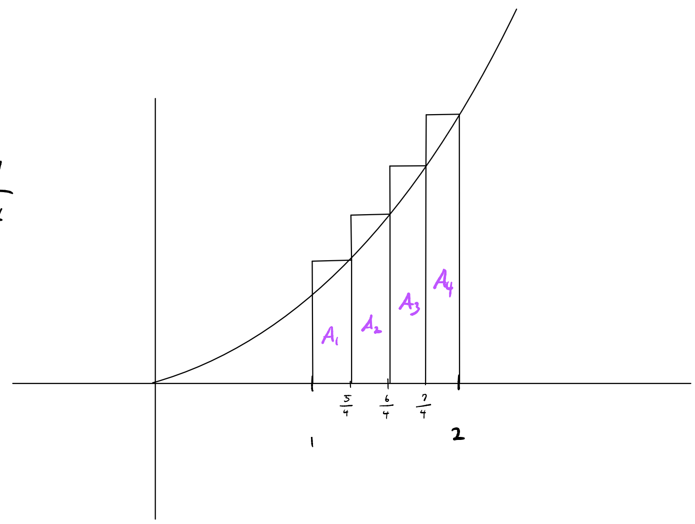
$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$

$x_1 = a + 1 \cdot \Delta x = 1 + 1 \cdot \frac{1}{4} = \frac{5}{4}$

$x_2 = a + 2 \cdot \Delta x = 1 + 2 \cdot \frac{1}{4} = \frac{6}{4}$

$x_3 = a + 3 \cdot \Delta x = 1 + 3 \cdot \frac{1}{4} = \frac{7}{4}$

$x_4 = a + 4 \cdot \Delta x = 1 + 4 \cdot \frac{1}{4} = \frac{8}{4}$



Approximate area is

$$\underbrace{f(x_1)\Delta x}_{A_1} + \underbrace{f(x_2)\Delta x}_{A_2} + \underbrace{f(x_3)\Delta x}_{A_3} + \underbrace{f(x_4)\Delta x}_{A_4}$$

$$= \Delta x \left(f(x_1) + f(x_2) + f(x_3) + f(x_4) \right)$$

$$= \boxed{\frac{1}{4}\left(\left(\frac{5}{4}\right)^2 + \left(\frac{6}{4}\right)^2 + \left(\frac{7}{4}\right)^2 + \left(\frac{8}{4}\right)^2 \right)}$$

4. Consider the functions

$$g(x) = \int_0^x t^2\, dt \qquad\qquad h(x) = \int_0^x \sin(t^3)\, dt$$

(a) What is the geometric meaning of the number $g(5)$?

$g(5) = \int_0^5 t^2\, dt$ which means the area underneath the curve t^2 on $[0,5]$.

(b) What is the geometric meaning of the number $h(3)$?

$h(3) = \int_0^3 \sin(t^3)\, dt$ which means the area underneath the curve $\sin(t^3)$ on $[0,3]$.

(c) Evaluate the following expression:

$$\frac{d}{dx}[2g(x) + 3h(x)]$$

$$= 2 \frac{d}{dx}[g(x)] + 3 \frac{d}{dx}[h(x)]$$

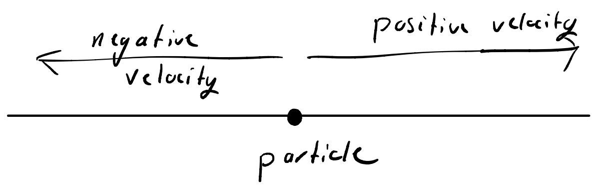
$$= 2 \frac{d}{dx} \int_0^x t^2\, dt + 3 \frac{d}{dx} \int_0^x \sin(t^3)\, dt$$

$$= \boxed{2x^2 + 3\sin(x^3)}$$ FTC 1

5. A particle is traveling along a horizontal line. The instantaneous velocity is

$$v(t) = t^2 - 2t - 3$$

(a) Draw a visualization of the particle with both negative and positive velocity.



(b) Determine the total displacement on the time interval $[2, 4]$. Did the particle move to the left or the right of the starting point?

$$\text{displacement} = \int_2^4 v(t)\,dt = \int_2^4 (t^2 - 2t - 3)\,dt = \frac{t^3}{3} - 2\frac{t^2}{2} - 3t \Big|_2^4$$

$$= \frac{t^3}{3} - t^2 - 3t \Big|_2^4 = \frac{4^3}{3} - 4^2 - 3\cdot 4 - \left(\frac{2^3}{3} - 2^2 - 3\cdot 2\right)$$

$$= \frac{64}{3} - 28 - \left(\frac{8}{3} - 10\right)$$

$$= \frac{64}{3} - \frac{8}{3} - 28 + 10$$

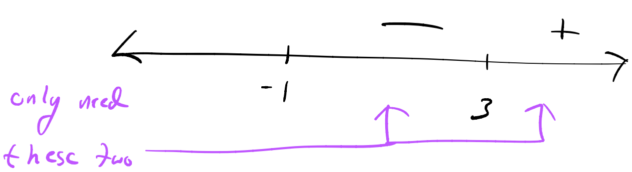
$$= \frac{56}{3} - \frac{54}{3}$$

$$= \boxed{\frac{2}{3}}$$

the particle moved right $\frac{2}{3}$ units from the starting position.

(c) Now determine the total distance traveled on $[2, 4]$.

Sign diagram of $v(t) = t^2 - 2t - 3 = (t-3)(t+1)$



$v(0) = (0-3)(0+1) = -$

$v(4) = (4-3)(4+1) = +$

So distance $= \displaystyle\int_2^4 |v(t)|\,dt$

$$= \int_2^3 |t^2 - 2t - 3|\,dt + \int_3^4 |t^2 - 2t - 3|\,dt$$

$$= \int_2^3 -(t^2 - 2t - 3)\,dt + \int_3^4 (t^2 - 2t - 3)\,dt$$

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$$= \int_2^3 (-t^2 + 2t + 3)\, dt \quad + \left[\frac{t^3}{3} - t^2 - 3t \right]_3^4$$

$$= -\frac{t^3}{3} + t^2 + 3t \Big|_2^3 + \frac{4^3}{3} - 4^2 - 3 \cdot 4 - \left(\frac{3^3}{3} - 3^2 - 3 \cdot 3 \right)$$

$$= -\frac{3^3}{3} + 3^2 + 3 \cdot 3 - \left(-\frac{2^3}{3} + 2^2 + 3 \cdot 2 \right) + \frac{64}{3} - 28 - (-9)$$

$$= 9 - \left(-\frac{8}{3} + 10 \right) + \frac{64}{3} - 19$$

$$= 9 - 10 - 19 + \frac{8}{3} + \frac{64}{3}$$

$$= -20 + \frac{72}{3}$$

$$= -20 + 24$$

$$= \boxed{4}$$

$\sin^2\theta = (\sin\theta)^2$ 6. Evaluate and fully simplify the following:

(a) $\displaystyle\int \sin^2\theta\cos\theta\, d\theta = \int u^2\, du = \frac{u^3}{3} + C = \boxed{\frac{\sin^3(\theta)}{3} + C}$

pick inside of composition

$u = \sin\theta$

$du = \cos\theta\, d\theta$

pick inside of composition

(b) $\displaystyle\int_0^{\sqrt{\pi}} x\cos(x^2)\, dx = \int_0^{\pi} \cos(u)\cdot\frac{1}{2}\, du$

$u = x^2$

$du = 2x\, dx$

$x\, dx = \frac{1}{2}\, du$

$\displaystyle = \frac{1}{2}\sin(u)\Big|_0^{\pi}$

$= \frac{1}{2}\sin(\pi) - \frac{1}{2}\sin(0)$

left: when $x = 0$, $u = 0$

right: when $x = \sqrt{\pi}$, $u = \pi$

$= \boxed{0}$

use precalculus to simplify

pick inside of composition

(c) $\int x^2\sqrt{x+1}\,dx = \int (u-1)^2 \sqrt{u}\, du$

$u = x+1$

$du = 1 \cdot dx$

Need x^2, solve for x

$x = u-1$

$x^2 = (u-1)^2$

$$= \int (u^2 - 2u + 1) \cdot u^{\frac{1}{2}}\, du$$

$$= \int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

$$= \frac{2}{7} u^{\frac{7}{2}} - 2 \cdot \frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \boxed{\frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + C}$$

(d) $\int_0^1 \cos\left(\frac{\pi t}{2}\right) dt = \int_0^{\frac{\pi}{2}} \cos(u) \cdot \frac{2}{\pi}\, du$

$u = \frac{\pi}{2} t$

$du = \frac{\pi}{2}\, dt$

$dt = \frac{2}{\pi}\, du$

left: when $t = 0$, $u = 0$

right: when $t = 1$, $u = \frac{\pi}{2}$

$$= \frac{2}{\pi} \sin(u) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi} \sin(0)$$

$$= \frac{2}{\pi} \cdot 1 - \frac{2}{\pi} \cdot 0$$

$$= \boxed{\frac{2}{\pi}}$$